Approximate ℓ-State Solutions of the Klein-Gordon Equation for Modified Woods-Saxon Potential With Position Dependent Mass

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Abstract

The radial part of the Klein-Gordon equation for the generalized Woods-Saxon potential is solved by using the Nikiforov-Uvarov method in the case of spatially dependent mass within the new approximation scheme to the centrifugal potential term. The energy eigenvalues and corresponding normalized eigenfunctions are computed. The solutions in the case of constant mass are also studied to check out the consistency of our new approximation scheme.

Keywords: Woods-Saxon Potential, Position Dependent Mass, Klein-Gordon Equation, Nikiforov-Uvarov Method

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I. INTRODUCTION

The investigation of the quantum mechanical systems in the case of position dependent mass (PDM) following of works by v. Roos, and Levy-Leblond [1, 2] have recently been received great attentions. This is so because such solutions are aviable in wide range of different areas, for example, in the study of impurities in crystals [3-5], or of electronic properties of quantum wells, and quantum dots [6], and in semiconductor heterostructures [7]. Yahiaowi, and Bentaiba [8] have studied the weak-pseudo-Hermiticity in the case of PDM, Ganguly, and Nieto have extended the second-order supersymmetric approach to the systems with coordinate dependence mass [9]. In Ref. [10], some new shape-invariant, exactly solvable potentials are generated by using a specific ansatz in the point of PDM-case. Ju et al. have been studied the dynamics of a quasi-free particle in an effective potential arising from the dependence of the mass on coordinates, and analyzed the eigenfunctions and probability densities for s-waves [11].

Another interesting area received a lot of attentions is that solving the Schrödinger (SE), and Dirac equations in the case of PDM. To solve the above equations has been used different methods, and approaches for different potentials, such as deformed algebras in Coulomb potential [12], in the content of supersymmetric quantum mechanics [13, 14, 15, 16], quadratic algebra approach [17], numerical analysis of a square potential by using appropriate matching conditions [18], point canonical transformation applying on harmonic oscillator, Coulomb and Morse class of potentials [19], finding the non-relativistic Green's functions with PDM for harmonic oscillator [20], Coulomb potential in Dirac equation [21], Morse potential in PDM background [22], a series solution of the SE for Cornell potential [23], finding the bound states of Rosen-Morse and Scarf potentials via general point canonical transformation [24].

In the present work, we give the approximate solutions, and corresponding wave functions of the radial Klein-Gordon (KG) equation for the Woods-Saxon (WS) potential in the case of PDM. We investigate the energy spectrum, and the corresponding eigenfunctions of the generalized WS potential by using a new approximation to the centrifugal potential. In order to find the spectrum we use the NU-method in the case of exponentially mass distribution varying with coordinate. The NU-method is a powerful tool to solve of the second order linear differential equations with special orthogonal functions. In this method, the differential equation is turned into a hypergeometric type equation by using a transformation on

coordinate [25].

The organization of this work is as follows. In Section II, we solve the radial part of the KG-equation for generalized WS potential by using the NU-method within the framework of an approximation to the centrifugal term. We compute the energy eigenvalues and corresponding eigenfunctions, and also give the results for the case of the constant mass to control the consistency of our new approximation. We write our conclusions in Section III.

II. NIKIFOROV-UVAROV METHOD AND CALCULATIONS

In spherical coordinates, the radial part of the Klein-Gordon equation can be written as [30]

$$\Big\{\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - \frac{\hbar^2\ell(\ell+1)}{2mr^2} - \frac{1}{2mc^2}[m^2c^4 - (E - V(r))^2]\Big\}\phi(r) = 0\,, \tag{1}$$

where ℓ is the angular-momentum quantum number, E is the energy of the particle, m is the rest mass, and c is the velocity of the light.

The generalized WS potential can be written of the form [31]

$$V(r) = -\frac{V_0}{1 + qe^{\beta(r-r_0)}}\,, \quad (0 \le r \le \infty)\,. \tag{2}$$

where V_0 is the potential depth, β is a short notation, i.e. $\beta \equiv 1/a$, a is diffuseness of the nuclear surface, r is the center-of-mass distance between the projectile and target nucleus, and r_0 is the width of the potential, which is proportional with target mass number A. q is the deformation parameter, and arbitrarily taken to be a real constant. The WS potential is widely used in the coupled-channels calculations in heavy-ion physics. This model explains the single-particle motion during a heavy-ion collisions [26-29].

Let us write the potential as

$$V(x) = -\frac{V_0}{1 + qe^{\beta x}}\,, \tag{3}$$

where $x = (r - r_0)$. Eq. (1) can not be solved exactly because of the centrifugal potential term for $\ell \neq 0$. The nuclear distance r can not fluctuate very far from the equilibrium for

rather high vibrational levels [32], which gives small x-values. So the centrifugal potential term can be expand about $x = 0$ as the following

$$V_1(r) = \frac{\ell(\ell+1)}{r^2} = \frac{D}{(1+\frac{x}{r_0})^2} = D(1 - 2\,(\frac{x}{r_0}) + 3\,(\frac{x}{r_0})^2 + \ldots)\,, \tag{4}$$

where the parameter D in the above equation is given as $D = \frac{\hbar^2\ell(\ell+1)}{2mr_0^2}$.

Instead, we suggest to replace $V_1(r)$ by the following potential form [33]

$$V_1'(x) = (1+qe^{\beta x})^{-2}\Big[DD_0(1+qe^{\beta x})^2 + DD_1(1+qe^{\beta x}) + DD_2\Big]\,, \tag{5}$$

where the parameters D_0, D_1, and D_2 are arbitrary constants.

Expanding the potential $V_1'(x)$ around $x = 0$ under the same condition, and than combining equal powers with Eq. (4), one can find the arbitrary constants $D_i (i = 0, 1, 2)$ in the new form of the potential as

$$D_0 = 1 - \frac{(1+q)^2}{\beta r_0 q^2}\left[-\frac{3}{\beta r_0} + 1\right], \tag{6}$$

$$D_1 = \frac{(1+q)^2}{\beta r_0 q^2}\left[-\frac{6(1+q)}{\beta r_0} + 3q - 1\right], \tag{7}$$

$$D_2 = \frac{(1+q)^3}{\beta r_0 q^2}\left[\frac{3(1+q)}{\beta r_0} + \frac{1-q}{2}\right]. \tag{8}$$

where it can be seen that the new parameters D_0, D_1, and D_2 are real, dimensionless parameters, and dependent to the numerical values of the quantum system under consideration.

On the other hand, we prefer to use the following position dependent mass function

$$m(x) = m_0\left[1 - \frac{m_1}{m_0}\left(1+qe^{\beta x}\right)^{-1}\right]\ (m_0 > m_1)\,, \tag{9}$$

where m_0 and m_1 are two arbitrary positive parameters. The mass function is finite at infinity, and enables us to solve analytically the KG-equation given by Eq. (1), and to check out the limit of the case of the constant mass.

Substituting Eq. (5), and (9) into Eq. (1), we get

$$\Big\{\frac{d^2}{dx^2} \;-\; \Big[\frac{1}{\hbar^2c^2}(m_0^2c^4-E^2)+\frac{\hbar^2\ell(\ell+1)}{r_0^2}D_0\Big] + \Big[\frac{2}{\hbar^2c^2}(EV_0+m_0m_1c^4)-\frac{\hbar^2\ell(\ell+1)}{r_0^2}D_1\Big]\frac{1}{1+qe^{\beta x}} + \Big[\frac{1}{\hbar^2c^2}(V_0^2-m_1^2c^4)-\frac{\hbar^2\ell(\ell+1)}{r_0^2}D_2\Big]\frac{1}{(1+qe^{\beta x})^2}\Big\}\phi(x)=0 \tag{10}$$

By using the transformation $z=2(1+qe^{\beta x})^{-1}$, we have

$$\frac{d^2\phi(z)}{dz^2}+\frac{2(1-z)}{z(2-z)}\frac{d\phi(z)}{dz}+\frac{1}{[z(2-z)]^2}\left[-a_1^2z^2-a_2^2z-a_3^2\right]\phi(z)=0\,. \tag{11}$$

where

$$\begin{aligned} a_1^2 &= \omega_1^2\ell(\ell+1)D_2+\omega_2^2(m_1^2c^4-V_0^2)\,,\\ a_2^2 &= 2[\omega_1^2\ell(\ell+1)D_1-2\omega_2^2(m_0m_1c^4+EV_0)]\,,\\ a_3^2 &= 4[\omega_1^2\ell(\ell+1)D_0+\omega_2^2(m_0^2c^4-E^2)]\,. \end{aligned} \tag{12}$$

and $\omega_1^2=1/\beta^2r_0^2$, and $\omega_2^2=1/\hbar^2c^2\beta^2$.

To apply the NU-method, we rewrite Eq. (11) in the following form

$$\phi''(z)+\frac{\tilde{\tau}(z)}{\sigma(z)}\phi'(z)+\frac{\tilde{\sigma}(z)}{\sigma^2(z)}\phi(z)=0, \tag{13}$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials with second-degree, at most, and $\tilde{\tau}(z)$ is a polynomial with first-degree. By using the following transformation for the total wave function

$$\phi(z)=\xi(z)\psi(z) \tag{14}$$

we get a hypergeometric type equation

$$\sigma(z)\psi''(z)+\tau(z)\psi'(z)+\lambda\psi(z)=0, \tag{15}$$

where $\xi(z)$ satisfies the equation

$$\xi'(z)/\xi(z) = \pi(z)/\sigma(z). \tag{16}$$

and the other part, $\psi(z)$, is the hypergeometric type function whose polynomial solutions are given by

$$\psi_n(z) = \frac{b_n}{\rho(z)} \frac{d^n}{dz^n}[\sigma^n(z)\rho(z)], \tag{17}$$

where b_n is a normalization constant, and the weight function $\rho(z)$ must satisfy the condition

$$\frac{d}{dz}[\sigma(z)\rho(z)] = \tau(z)\rho(z). \tag{18}$$

The function $\pi(z)$ and the parameter λ required for this method are defined as follows

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{(\frac{\sigma'(z) - \tilde{\tau}(z)}{2})^2 - \tilde{\sigma}(z) + k\sigma(z)}\,, \tag{19}$$

$$\lambda = k + \pi'(z) \tag{20}$$

The constant k is determined by imposing a condition such that the discriminant under the square root should be zero. Thus one gets a new eigenvalue equation

$$\lambda \;=\; \lambda_n = -n\tau' - \frac{n(n-1)}{2}\,\sigma''\,, (n = 0, 1, 2, \ldots) \tag{21}$$

where

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z)\,. \tag{22}$$

and the derivative of $\tau(z)$ must be negative.

Comparing Eq. (11) with Eq. (13), we have

$$\tilde{\tau}(z) = 2(1-z)\,, \quad \sigma(z) = z(2-z)\,, \quad \tilde{\sigma}(z) = -a_1^2 z^2 - a_2^2 z - a_3^2 \tag{23}$$

Substituting this into Eq. (19), we get

$$\pi(z) = \pm\sqrt{(a_1^2 - k)z^2 + (a_2^2 + 2k)z + a_3^2}. \tag{24}$$

The constant k can be determined by the condition that the discriminant of the expression under the square root has to be zero

$$(a_2^2 + 2k)^2 - 4a_3^2(a_1^2 - k) = 0\,. \tag{25}$$

The roots of k are $k_{1,2} = -\frac{1}{2}\,a_2^2 - \frac{1}{2}\,a_3^2 \mp \frac{1}{2}\,a_3 A$, where $A = \sqrt{a_3^2 + 2a_2^2 + 4a_1^2}$. Substituting these values into Eq.(19), we get for $\pi(z)$ for k_1

$$\pi(z) = \mp\Big[\Big(\frac{A}{2} - \frac{a_3}{2}\Big)z + a_3\Big]\,, \tag{26}$$

and for k_2

$$\pi(z) = \mp\Big[\Big(\frac{A}{2} + \frac{a_3}{2}\Big)z - a_3\Big]\,, \tag{27}$$

Now we find the polynomial $\tau(z)$ from $\pi(z)$ for the second choice as

$$\tau(z) = 2 + 2a_3 - 2\Big(\frac{A}{2} + \frac{a_3}{2} + 1\Big)z. \tag{28}$$

so its derivative $-2\Big(\frac{A}{2} + \frac{a_3}{2} + 1\Big)$ is negative. We have from Eq. (20)

$$\lambda = -\frac{1}{2}\left(a_2^2 + a_3^2 + Aa_3 + A + a_3\right), \tag{29}$$

and Eq. (21) gives us

$$\lambda_n = 2n\Big(\frac{A}{2} + \frac{a_3}{2} + 1\Big) + n^2 - n\,. \tag{30}$$

Substituting the values of the parameters given by Eq. (12), and setting $\lambda = \lambda_n$, one can find the energy eigenvalues for any ℓ-states

$$E_{n,\ell} = -\frac{V_0[4\omega_2^2V_0^2+N^2-4\omega_2^2\ell(\ell+1)(D_1+D_2)+4m_1c^4\omega_2^2(2m_0-m_1)]}{2(N^2+4\omega_2^2V_0^2)} \pm \frac{N}{\omega_2}\sqrt{\frac{\omega_1^2(2D_0+D_1+D_2)\ell(\ell+1)+2m_0^2c^4\omega_2^2}{2(N^2+4\omega_2^2V_0^2)}-\left[\frac{\omega_1^2(D_1+D_2)\ell(\ell+1)}{N^2+4\omega_2^2V_0^2}\right]^2-\frac{1}{16}+\tilde{m}_1^2}\,, \tag{31}$$

where the energy eigenvalues with (+) sign correspond to particle, and the one with (-) sign correspond to antiparticle. Two parameters in the above expression are

$$\tilde{m}_1 = \frac{\sqrt{8m_1c^4\omega_2^2(2m_0-m_1)[4\omega_1^2(D_1+D)\ell(\ell+1)-2\omega_2^2m_1c^4(2m_0-m_1)-(N^2+4\omega_2^2V_0^2)]}}{N^2+4\omega_2^2V_0^2}\,, \tag{32}$$

and

$$N = -(2n+1)+\sqrt{1+4a_1^2}\,. \tag{33}$$

We see that the energy levels for particles and antiparticles are symmetric, and the ground state energy is different from zero. We summarize some numerical results in Table I to see the effect of the spatially dependent mass parameter m_1 on the energy eigenvalue of bound states. It is observed that the energy levels are strongly dependent on the parameter, and the increase of the energy eigenvalues in the existence of m_1 is very significant. It has to be stress that the higher numerical values of the parameter m_1 give positive values for the bound states.

The energy spectra in the case of constant mass is obtained by setting $m_1 = 0$ in Eq. (32) which gives us $\tilde{m}_1 = 0$, and we get

$$E_{n,\ell}^{m_1=0} = -\frac{V_0[4\omega_2^2V_0^2+N^2-4\omega_2^2\ell(\ell+1)(D_1+D_2)]}{2(N^2+4\omega_2^2V_0^2)} \pm \frac{N}{\omega_2}\sqrt{\frac{\omega_1^2(2D_0+D_1+D_2)\ell(\ell+1)+2m_0^2c^4\omega_2^2}{2(N^2+4\omega_2^2V_0^2)}-\left[\frac{\omega_1^2(D_1+D_2)\ell(\ell+1)}{N^2+4\omega_2^2V_0^2}\right]^2-\frac{1}{16}}\,, \tag{34}$$

where

$$N' = -(2n+1)+\sqrt{1+4a_1'^2}\,,\;\; a_1'^2 = a_1^2(m_1\to 0). \tag{35}$$

It is seen that the result for the case of constant mass is the same with those obtained in Ref. (29).

In order to find the eigenfunctions, we first compute the weight function from Eq. (18)

$$\rho(z) = z^{a_3}(2-z)^{A}\,, \tag{36}$$

and the wave function becomes

$$\psi_{n\ell}\,(z) \;=\; \frac{b_n}{z^{a_3}(2-z)^{A}}\,\frac{d^n}{dz^n}\,\Big[\,z^{n+a_3}\,(2-z)^{n+A}\Big]\,. \tag{37}$$

where b_n is a normalization constant. The polynomial solutions can be written in terms of the Jacobi polynomials [34, 35]

$$\psi_{n\ell}\,(z) = b_n\,P_n^{(a_3,\,A)}\,(1-z)\,, \quad A>-1\,, \quad a_3>-1\,. \tag{38}$$

On the other hand, the other part of the wave function is obtained from Eq. (16) as

$$\xi(z) = z^{a_3/2}\,(2-z)^{A/2}\,. \tag{39}$$

Thus, the total eigenfunctions take

$$\phi_{n\ell}\,(z) = b'_n\,(2-z)^{A/2}z^{a_3/2}\,P_n^{(a_3,\,A)}\,(1-z)\,. \tag{40}$$

where b'_n is the new normalization constant. It is obtained from

$$\frac{\beta}{4}\int_0^1 |\phi_{n\ell}(z)|^2\,\Big(\,\frac{z-2}{z}\,\Big)dz = 1\,. \tag{41}$$

To evaluate the integral, we use the following representation of the Jacobi polynomials [35]

$$\begin{aligned} P_n^{(\sigma,\varsigma)}(z) \;&=\; \frac{\Gamma(n+\sigma+1)}{n!\Gamma(n+\sigma+\varsigma+1)} \\ &\times\;\sum_{m=0}^{n}\begin{pmatrix} n \\ r \end{pmatrix}\frac{\Gamma(n+\sigma+\varsigma+m+1)}{\Gamma(m+\sigma+1)}\,\frac{\Gamma(n+\sigma+\varsigma+m+1)}{\Gamma(m+\sigma+1)}\,(\,\frac{z-1}{2})^m\,, \end{aligned} \tag{42}$$

where $\left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$. Hence, from Eq. (41), and with the help of Eq. (42), we get

$$[g(n,m) \times g(r,s)]\Big(\frac{\beta}{4}\Big)\,|b'_n|^2 \int_0^1 z^{m+s+a_3-1}\,(2-z)^{A+1}\,dz = 1\,, \tag{43}$$

where $g(n,m)$, and $g(r,s)$ are two arbitrary functions of the parameters A, and a_3, and given by

$$\begin{aligned} g(n,m) \;&=\; \frac{2^{-m}\,\Gamma(A+n+1)}{n!\Gamma(A+a_3+n+1)} \\ &\times\; \sum_{m=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) \frac{\Gamma(n+\sigma+\varsigma+m+1)}{\Gamma(m+\sigma+1)}\,\frac{\Gamma(A+a_3+n+m+1)}{\Gamma(a_3+m+1)}\,(-1)^m\,, \end{aligned} \tag{44}$$

and

$$g(r,s) = g(n,m)\,(n \to r; m \to s)\,. \tag{45}$$

The integral in Eq. (43) can be evaluated by using the following integral representation of hypergeometric type function ${}_2F_1(a,b;c;z)$ [36]

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-tz)^{-a}\,dt\,, \tag{46}$$

by setting the variable $z \to \frac{z}{2}$, and taking $c = 1+b\,, z = 1$, one gets

$$\int_0^1 t^{b-1}\,(2-t)^{-a}\,dt = \frac{\Gamma(b)\Gamma(1)}{2^a\,\Gamma(1+b)}\,{}_2F_1\,(a,b;1+b;\,\frac{1}{2}\,)\,, \tag{47}$$

From last equation

$$\begin{aligned} \int_0^1 z^{m+s+a_3-1}\,(2-z)^{A+1}\,dz \;&=\; \frac{\Gamma(m+s+a_3)\Gamma(1)}{2^a\,\Gamma(m+s+a_3+1)} \\ &\times\; {}_2F_1(-A-1, m+s+a_3; m+s+a_3+1;\,\frac{1}{2}\,)\,, \end{aligned} \tag{48}$$

where we set $b = m + s + a_3$, and $a = -A - 1$.

By using the following identities of hypergeometric type functions [36]

$$ {}_2F_1\left(a, b; c; -1\right) \;=\; \frac{\Gamma\Big(\frac{1}{2}b+1\Big)\Gamma(b-a+1)}{\Gamma(b+1)\Gamma\Big(\Big(\frac{1}{2}b-a+1\Big)}\,, \;\; (a-b+c=1\,,\; b>0) \tag{49} $$

$$ {}_2F_1\left(a, b; c; \frac{1}{2}\right) \;=\; 2^a\, {}_2F_1\left(a, c-b; c; -1\right), \tag{50} $$

the function of ${}_2F_1\left(a, b; c; z\right)$ in Eq. (48) can be evaluated as

$$ {}_2F_1\left(-A-1, b; 1+b; \frac{1}{2}\right) = \; 2^{-A-2}\,\sqrt{\pi}\,\frac{\Gamma(A+3)}{\Gamma\Big(\frac{5}{2}\;+A\Big)}\,, \;\; (m+s+a_3-A=2)\,, \tag{51} $$

Finally, we get the normalization constant as

$$ |b'_n|^2 = \; \frac{8}{\beta\sqrt{\pi}}\,\frac{\Gamma(m+s+a_3+1)\Gamma\Big(\frac{5}{2}\;+A\Big)}{\Gamma(m+s+a_3)\Gamma(3+A)[g'(n,m)\times g(r,s)]}\,. \tag{52} $$

where

$$ \begin{aligned} g'(n,m) \;&=\; \frac{2^{-m}\,\Gamma(A+n+1)}{n!\Gamma(A+a_3+n+1)} \\ &\times\; \sum_{m=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right)\frac{\Gamma(n+\sigma+\varsigma+m+1)}{\Gamma(m+\sigma+1)}\,\frac{\Gamma(A+a_3+n+m+1)}{\Gamma(a_3+m+1)}\,(-1)^{m+1}\,. \end{aligned} \tag{53} $$

III. CONCLUSION

We have solved the radial part of the KG-equation for the modified Woods-Saxon potential in the case of position dependent mass by using a new approximation scheme to the centrifugal potential term for any ℓ values. It is observed that the results obtained by using the new scheme for the case of the constant mass are consistent with the ones obtained in Ref. [29]. It is seen that there is a linear relation between the energy eigenvalues and the contributions coming from the dependence of the mass on spherical coordinate. The energy spectra and the corresponding wave functions are obtained by applying the NU-method. The eigenfunctions can be expressed in terms of Jacobi polynomials in the scheme of the new approximation to the centrifugal barrier in the case of position dependent mass.

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TABLE I: The dependence of the bound states for a system 'proton+nucleon with average mass number $A = 56$' on the parameter m_1 in MeV for $q = 1$ by using the numerical values $m_p = 1.007825$ amu, $V_0 = 47.78$ MeV, $r_0 = 4.91623$ fm [37].

$m_1(amu)$	n	ℓ	$E_{n\ell} < 0$
0	0	0	171.920
	1	0	922.962
		1	924.286
	2	0	891.947
		1	895.473
		2	902.084
0.01	0	0	270.028
	1	0	842.200
		1	846.735
	2	0	808.765
		1	813.490
		2	822.663
0.001	0	0	187.762
	1	0	915.806
		1	917.461
	2	0	844.123
		1	887.762
		2	894.605